Fibonacci numbers when counting chord diagrams

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1. **Introduction**. The combinatorics of drawing some chords in some circles arises in a section of knot theory called invariants of finite-type. We will see that the Fibonacci numbers are useful for counting certain forms of these chord diagrams. We also prove that most of the chord diagrams are null.

2. **Definitions** We begin with a vector space of chord diagrams. At first glance, the vectors are deceptively simple because chord diagrams are circles oriented counter-clockwise with chords drawn in. The chord locations are considered equal up to diffeomorphism, which means we only pay attention to the relative locations of the endpoints of the chords, not to the lengths of chords or the shapes of chords. We can give the properties of the vector space quickly: the vectors are formal linear combinations of chord diagrams, modulo the 4-term and 1-term relations. We introduce these relations below; only the 1-term relation is central to our considerations. The 1-term relation says that any chord diagram with a chord not crossed by any other chords is a null vector.

The 4-term relation has a pair of chord diagrams on each side with changes in the positions of two chords, the remaining chords staying unchanged. The example below has two applications of the 4-term relation where the thicker chords are the ones which change relative to each other. The first diagram on each line are an inverse pair. The other terms on each line match, which implies the first two diagrams are the same.

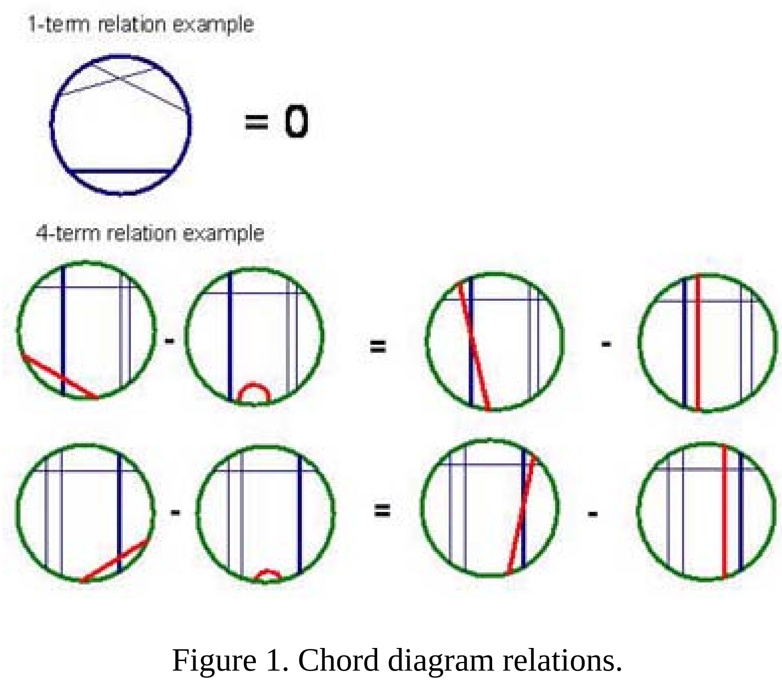

Figure 1. Chord diagram relations.

The vector space of chord diagrams is often referred to as A, with subspaces $A_2, A_3, \ldots A_m$, where m is the number of chords. The summary of its properties in [1] remains an essential source. We have also relied on the preprint of the text from Chmutov, Duzhin and Mostovoy [2] for its use of current vocabulary and summary of previous results. Some big questions remain open regarding this vector space. For example, the dimension of the subspace A_m is unknown in general, mostly because the 4-term relation brings a complexity which is difficult to control. So far, all chord diagrams studied have been found equal to their inverses (for low m,) where the *inverse* of a chord diagram is the original diagram with its orientation reversed. In general, however, it remains unknown whether this property is true or not for all chord diagrams in the space.

In our summer research, we searched for families of self-invertible chord diagrams. That is, we wanted to establish some structures which were self-invertible. We started with chord diagrams from an object called a *wheel*.

Some linear combinations of chord diagrams are so useful that they are represented with figures called closed Jacobi diagrams.We illustrate below the STU relation which shows how a subtraction of chord diagrams is drawn as a figure with tri-valent points inside the outer loop. The arrows mean that the rest of the diagram is the same for all figures in the equation. Our vocabulary emulates the structure: the segment joining the paired arrows in the first circle is called a *spoke*. When the closed Jacobi diagram has a closed circle inside with spokes connecting the inner circle to the outer circle, we call the inner circle, the *hub*. The subscript illustrates how we will label a single spoke and name the arc between these spokes with the same label.

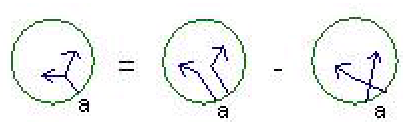

Figure 2. STU with labels.

To sum up, we now have a figure to go with the extremely suggestive name, *wheel*. The wheel illustrated below has p spokes. It stands for a sum of signed chord diagrams where each spoke has split into its two terms. A term gets a negative sign when it has an odd number of crossed, or transposed, endpoints from the second term of the STU. Any of the neighboring chords after the arrow could be transposed; we just drew the case with no transpositions.

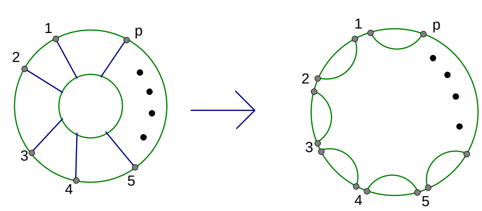

Figure. 3. Closed Wheel Diagram.

3. **Counting null chord diagrams**. Let's consider the resolution of all the triple points at once, through their corresponding endpoints, keeping the labels for pairs of endpoints which could be resolved the other way by the STU. We will refer to a chord by its lower-numbered endpoint, with the exception of the chord with endpoints p and 1, which we shall call c_p. That way we have a name for each chord numbered in the natural way. When we need to discuss arcs, we will refer to the arc between chord endpoints by this same rule.

If the closed wheel Jacobi diagram is rewritten as a linear combination of chord diagrams, we discover that many of the chord diagrams have at least one isolated chord and therefore are null. (When chord diagrams are null, they are self-inverse diagrams.) We are going to count the number of null chord diagrams in the STU decomposition of the closed wheel diagram. With p spokes, the closed wheel diagram decomposes into 2^p chord diagrams.

We designate some chord c_k. We will count the diagrams such that every chord preceding c_k is not isolated, and every chord after c_k can be isolated or not be isolated. To isolate chord c_k, we cannot transpose endpoints at k or endpoints at $k+1$. However, we must transpose endpoints at $k-1$ in order to keep chord c_{k-1} not isolated.

So, to keep the chords before c_k not isolated, some of their endpoints must be transposed, but not necessarily all of them. This brings about the need for a set of optionally transposed endpoints, which begin at 1 *or* 2 *or both*, *and*, 2 *or* 3 *or both*, *and*,..., *and* $k-3$ or $k-2$ or both, which we will call the set H_k. This structure will remind number theory fans of a famous sequence.

Lemma $|H_k| = F_k$, *where* F_k *is the* k^{th} *Fibonacci number*.

Proof To isolate c_1 the set of transpositions, α_1, must not include endpoints at 1 or 2. Endpoints beyond c_1 are not included in α_1. Thus, we have no optional endpoints to include in α_1, so the set of optionally transposed endpoints $H_1 = \emptyset$ and $|H_1| = 1$. This leaves $p-2$ endpoints remaining above c_1, so there are 2^{p-2} ways to isolate chord 1. (We might as well count our chord diagrams while we work the proof.)

To isolate c_2 with c_1 not isolated, the set of transpositions, α_2 must not include endpoints at 2 or 3, but must include endpoints at 1. So, again there are no open choices to put in α_2. The only choice is to choose nothing...

$$H_2 = \emptyset \qquad |H_2| = 1$$

There are 2^{p-3} ways to isolate c_2 without isolating c_1.

To isolate c_3 without isolating c_1 and c_2, the set of transpositions must not contain endpoints at 3 or 4, but must contain endpoints at 2. Since the endpoints at 2 are transposed, chord 1 is automatically not isolated. Thus, our first open choices are the endpoints at 1, so $H_3 = \{\{1\},\emptyset\}$. There are 2^{p-4} options for the endpoints numbered above c_3 and $|H_3| = 2$.

To isolate c_4 with chords 1, 2, and 3 not isolated, the set α_4 must not contain the endpoints at 4 and 5 but must contain 3. This gives the endpoints below c_3 options to be included in α_4. We write them as 1 or 2 or both; $H_4 = \{\{1\},\{2\},\{1,2\}\}$ and $|H_4| = 3$. There are 2^{p-5} options beyond chord 4.

To isolate c_5 with chords 1, 2, 3 and 4 not isolated, the set α_5 must not contain endpoints at 5 or 6 but must contain endpoints at 4. The optional endpoints at 1, 2, 3 may be stated as 1 or 2 or both, and, 2 or 3 or both. Note that we can build H_5 from H_4 and H_3 by appending a 3 to each set in H_4 and a 2 to each set in H_3. This gives $H_5 = \{\{1,3\},\{2,3\},\{1,2,3\},\{1,2\},\{2\}\}$ and $|H_5| = 5$. The coefficient for the options beyond chord 5 is 2^{p-6}.

Besides establishing the beginnings of the Fibonacci sequence, we have also illustrated how to apply the induction assumption. We now consider H_{k+1}. We build H_{k+1} from H_k and H_{k-1}, as in the previous case. Remember, H_k leaves no chord from c_1 through c_{k-1} isolated, and H_{k-1} leaves no chord from c_1 through $k-2$. To isolate c_{k+1}, without isolating any previous chord, including endpoints at k in T_{k+1} is mandatory. Appending the coefficient $k-1$ to all sets in H_k gives a set of sets which leaves no chord isolated before c_{k+1} and a transposition at endpoints $k-1$. Appending the coefficient $k-2$ to all sets in H_{k-1} gives a set of sets which leaves no chord before c_k isolated and no transposition at endpoints $k-1$. The new appended sets are mutually exclusive.

To see that the union is exactly H_{k+1}, let $h \in H_{k+1}$. Then, h leaves no chord isolated for c_1 through c_k, when a transposition at k is automatic. Thus, h has 1 or 2 or both, and, 2 or 3 or both, and so on up to $k-2$ or $k-1$ or both. If $k-1 \in h$, then h is an element of the appended version of H_k. If $k-1 \notin h$, then h is an element of the appended version of H_{k-1}.

So $|H_{k+1}| = |H_k| + |H_{k-1}| = F_k + F_{k-1} = F_{k+1}$. ■

We can use this formula, with its coefficients, to count the number of null chord diagrams in the closed wheel diagram. We add together all the cases we have considered, including the case H_{p-1}. But, the case c_p, the last step, is an exception because there is no numbered chord above it. Let's count the number of chord diagrams with only chord p isolated. Then α_p must contain the endpoints at $p-1$ and at 2. Then the options are at 3 or 4 or both, and 4 or 5 or both, and... so on up to $p-3$ or $p-2$ or both. Luckily, we're now experts at counting this: F_{p-2}.

The above proof is longer than necessary for those readers familiar with the various ways the Fibonacci numbers occur in counting problems. We were unfamiliar with this Fibonacci structure and we were amazed to see our problem's approximate structure rewritten on Fibonacci puzzle pages. Our Fibonacci reference [3] gives as close a version as anybody else's.

So, the number of null chord diagrams for the closed wheel diagram with p spokes is

$$F_{p-2} + \sum_{i=1}^{p-1} F_i 2^{p-1-i}.$$

After calculating some values for this number, we noticed that it grows almost as fast as 2^p. We programmed our calculators to find the ratio between our sum and 2^p for $p < 45$ and the ratio approached 1. We went to work on the limit of this ratio and found the ratio of null diagrams to total diagrams does indeed approach 1 as p increases, with a proof worth sharing with people outside knot theory.

Theorem $$\lim_{p\to\infty}\left(\frac{1}{2^p}\left(F_{p-2} + \sum_{i=1}^{p-1} F_i 2^{p-1-i} \right)\right) = 1$$

Proof. Since there are 2^p chord diagrams when we take all possible resolutions of spokes, we now have the following inequalities. (Dropping the F_{p-2} term from the middle expression felt overly generous at first, because F_{p-2} gets large as p increases. But the move turned out so well that we might call this sacrifice, "golden.")

$$\sum_{i=1}^{p-1} F_i 2^{p-1-i} \le \sum_{i=1}^{p-1} F_i 2^{p-1-i} + F_{p-2} \le 2^p.$$

There is a Fibonacci formula using the Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$. It states $F_n = \dfrac{\varphi^n - (\frac{-1}{\varphi})^n}{\sqrt{5}}$ [4], which turns the left-hand sum into a difference of two geometric series, when we divide by 2^p. Taking the limit as $p \to \infty$ squeezes the ratio of null chord diagrams to 1. The interesting math happens in the left-hand limit. Here, we apply the formula $\varphi + 1 = \varphi^2$ several times in order to simplify, after taking the limit.

$$\lim_{p\to\infty} \frac{1}{2^p} \sum_{i=1}^{p-1} F_i 2^{p-1-i} = \lim_{p\to\infty} \frac{1}{2\sqrt{5}} \sum_{i=1}^{p-1} \frac{\varphi^i - (\frac{-1}{\varphi})^i}{2^i} =$$

$$\frac{1}{2\sqrt{5}}\left(\frac{\varphi}{2-\varphi} + \frac{1}{2\varphi+1}\right) = \frac{1}{2\sqrt{5}}(\varphi^3 + \varphi^{-3}) =$$

$$\frac{1}{2\sqrt{5}}(\varphi + \varphi^{-1})(\varphi^2 - 1 + \varphi^{-2}) = \frac{1}{2\sqrt{5}}(2\varphi - 1)(\varphi^2 - 1 + 2 - \varphi) =$$

$$\frac{1}{2\sqrt{5}}(2\varphi - 1)(\varphi + 2 - \varphi) = \frac{1}{\sqrt{5}}\left(2\frac{1+\sqrt{5}}{2} - 1\right) = 1 \qquad \blacksquare$$

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