Fibonacci numbers when counting chord diagrams

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1. Introduction. If we draw chords in circles so that the chords have the option of intersecting only neighboring chords, counting such arrangements uses the Fibonacci numbers, defined as $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 2$. We will show that most of these diagrams have a chord which does not intersect its neighbors.

2. Definitions A *chord diagram* is a circle, oriented counter-clockwise, with chords drawn in. Diagrams are considered equal up to diffeomorphism; this means the sizes or shapes of the chords do not matter. We pay attention to the relative position of the endpoints. So a two-chord diagram with chords that must cross is the same, whether the chords are two diameters or two small chords jammed into a quadrant of the circle. For knot theoretical reasons explained in the last section, we will focus on diagrams with a specific structure.

This structure has a physical example. Suppose a circular room has people standing against the wall, all the way around. The people must keep their backs to the wall. Each may slide a left foot to the left or a right foot to the right, past a neighbor's foot to the left or right (or both) so that intersections occur. So nobody can have both feet between another person's feet. Their trousers are like our chords. A chord may cross one or both of its neighbors or have no crossings at all. Also, no chord may be nested with both its endpoints between the endpoints of another chord. (We must allow an exception to this rule for the diagram with only 2 chords, no crossings, because two people standing next to each other in a round room don't *really* have their own feet between the other's feet.) We give each chord a name, starting with C_1 at the top of the circle, counting counter-clockwise. The endpoints of chord C_i are both called c_i because they will actually stand for the same point, as we shall see in the last section. We will label the arc of the circle between endpoints of adjacent chords using the higher subscript of the two neighbors, with the exception of the place where the first and last chords meet. There, we use the lower subscript in order to get a 1 in the logical spot.

We can now be very specific about the chord diagrams in this paper. When reading along the circle counter-clockwise, the first appearance of endpoint c_i may only have the second appearance of c_{i-1}, the first appearance of c_{i+1}, or the second appearance of c_i immediately following it. When chord C_i has its endpoints c_i consecutive, we call C_i an *isolated chord.* (When $i = p$, we use 1 instead of $p + 1$ in these rules.) In Figure 1, we have thick arcs for the alternating arcs between neighbors. A closed vertex path exists when we start at an endpoint, follow the thick arc to its other endpoint, then follow the chord at that endpoint to the chord's other endpoint and so on. A closed vertex path using alternating arcs and all the chords is always possible with our special chord diagrams. We will call our diagrams *one-piece diagrams.* Not all chord diagrams are one-piece and plenty of diagrams with the closed vertex path do not have our restricted structure. We will call any diagram with an isolated chord a *null chord diagram* for a reason given in the last section. We will call a place where two chords cross a *transposition* because if c_2 crosses c_3, an endpoint of c_3 appears between the endpoints of c_2.

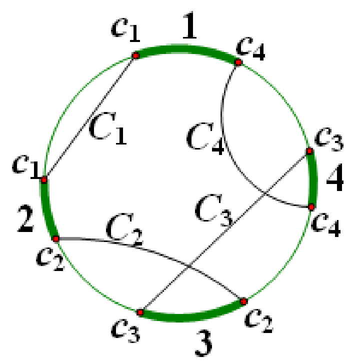

Figure 1. One-piece diagram.

3. Counting null chord diagrams. Let's consider the set D_p of all one-piece diagrams with p distinct chords. We are treating chords as distinctly numbered. Because we either have two consecutive chords crossing or not, and there are p places where endpoints are neighbors, the size of D_p is 2^p. We will now count the number of null chord diagrams in D_p.

We isolate some chord C_k. We will count the diagrams such that no chord preceding C_k is isolated, and every chord after C_k can be isolated or not be isolated. To isolate chord C_k, we cannot transpose endpoints at k or endpoints at $k+1$. However, we must transpose endpoints at $k-1$ in order to keep chord C_{k-1} not isolated.

To keep the chords before C_k not isolated, some of their endpoints must be transposed, but not necessarily all of them. We can name the arcs where transpositions could happen with the phrase *1 or 2 or both, and, 2 or 3 or both, and,..., and* $k-3$ or $k-2$ or both. (We have to handle $k=p$ separately, after the proof.) We will call the set of all optional choices which leave no chord before C_k isolated the set H_k. Each element of H_k is a subset of $\{1,2,..,p\}$ because each H_k is a set of subsets of transpositions.The number i is in one of these subsets if and only if the endpoints of C_{i-1} and C_i are transposed, (except for $i=1$, when C_1 and C_p have endpoints transposed.) We will use the notation $|H_k|$ for the number of elements in H_k. This structure will remind number theory fans of a famous sequence.

Lemma. $|H_k| = F_k$, *for* $k < p$, *where* F_k *is the* k^{th} *Fibonacci number.*

Proof. To isolate C_1, the set of transpositions H_1 must not include endpoints at arcs 1 or 2. Endpoints beyond C_1 are not included in H_1. Thus, we have no optional endpoints to include in H_1, so the set of chord diagrams with optionally transposed endpoints $H_1 = \{\emptyset\}$ and $|H_1| = 1$. This leaves $p - 2$ endpoints remaining above C_1, so there are 2^{p-2} ways to isolate chord 1. (We might as well count our chord diagrams while we work the proof.)

To isolate C_2 with C_1 not isolated, the set of transpositions H_2 must not include endpoints at arcs 2 or 3, but must transpose endpoints at 1. So, again there are no options to put in H_2 because the transposition at 1 is mandatory. So the set $H_2 = \{\emptyset\}$ and $|H_2| = 1$. (We have just begun our sequence of Fibonacci numbers with 1, 1.) There are 2^{p-3} ways to isolate C_2 without isolating C_1.

To isolate C_3 without isolating C_1 and C_2, the set of transpositions must not contain endpoints at 3 or 4, but must contain endpoints at 2. Since the endpoints at 2 are transposed, C_1 is automatically not isolated. Thus, our first option is to transpose, or not transpose, the endpoints at arc 1, so $H_3 = \{\{1\}, \emptyset\}$. There are 2^{p-4} options for the endpoints numbered above C_3 and $|H_3| = 2$.

To isolate C_4 with chords C_1, C_2, and C_3 not isolated, the set H_4 must not contain the endpoints at arcs 4 and 5 but must contain 3. This gives the arcs below C_3 options to be included in $H_{4.}$ We write them as 1 or 2 or both; $H_4 = \{\{1\}, \{2\}, \{1,2\}\}$ and $|H_4| = 3$. There are 2^{p-5} options beyond chord C_4.

To isolate C_5 with chords C_1, C_2, C_3 and C_4 not isolated, the set H_5 must not contain endpoints at arcs 5 or 6 but must contain endpoints at 4. The optional endpoints at 1, 2, 3 may be stated as 1 or 2 or both, and, 2 or 3 or both. We note that we can build H_5 from H_4 and H_3 by appending a 3 to each set in H_4 and a 2 to each set in H_3. This gives $H_5 = \{\{1,3\}, \{2,3\}, \{1,2,3\}, \{1,2\}, \{2\}\}$and $|H_5| = 5$. The coefficient for the options beyond chord 5 is 2^{p-6}.

Besides establishing the beginnings of the Fibonacci sequence, we have also illustrated how to apply the induction assumption. We now consider H_{k+1}. We build H_{k+1} from H_k and H_{k-1}, as in the example. Remember, H_k leaves no chord from C_1 through C_{k-1} isolated, and H_{k-1} leaves no chord isolated from C_1 through C_{k-2}. To isolate C_{k+1}, without isolating any previous chord, we must transpose at arc k, so k is not listed in H_{k+1}. Appending $k - 1$ to all sets in H_k gives a set of sets which leaves no chord isolated before C_{k+1} and a transposition at endpoints $k - 1$. Appending $k - 2$ to all sets in H_{k-1} gives a set of sets which leaves no chord before C_k isolated and no transposition at arc $k - 1$. But that's all right because of the mandatory transposition at k. The new appended sets are mutually exclusive.

To see that the union of appended sets is exactly H_{k+1}, let $h \in H_{k+1}$. Then, h leaves no chord isolated for C_1 through C_k, when a transposition at k is automatic. Thus, h has 1 or 2 or both, and, 2 or 3 or both, and so on up to $k - 2$ or $k - 1$ or both. If $k - 1 \in h$, then h is an element of the appended version of H_k. If $k - 1 \notin h$, then h is an element of the appended version of H_{k-1}.

So $|H_{k+1}| = |H_k| + |H_{k-1}| = F_k + F_{k-1} = F_{k+1}$. $\square$

The cases through H_{p-1} all have a power of 2 times a corresponding Fibonacci number, except H_p, which has no numbered chord above it. Our Lemma does not apply to H_p. To finish our cases, let's count the number of chord diagrams with only chord p isolated. Then H_p must contain the endpoints at arcs $p-1$ and at 2, (Figure 2.) The options are at 3 or 4 or both, and, 4 or 5 or both, and... so on up to $p-3$ or $p-2$ or both; in other words, we have two less chords than usual to get crossed. Luckily, we're now experts at counting this: F_{p-2}. We can use these results to count the number of null chord diagrams out of the 2^p possible one-piece diagrams.

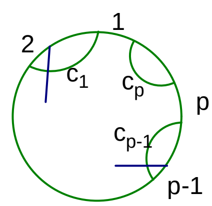

Figure 2. C_p.

The above proof is longer than necessary for those readers familiar with the various ways the Fibonacci numbers occur in counting problems. We were unfamiliar with this Fibonacci structure and we were amazed to see our problem's approximate structure rewritten on Fibonacci puzzle pages. Our Fibonacci reference [3] gives as close a version as anybody else's.

So, the number of null, one-piece diagrams with p chords is

$$F_{p-2} + \sum_{i=1}^{p-1} F_i 2^{p-1-i}.$$

After calculating some values for this number, we noticed that it grows almost as fast as 2^p. We programmed our calculators to find the ratio between our sum and 2^p for $p < 45$ and the ratio approached 1. We went to work on the limit of this ratio and found the ratio of null diagrams to total diagrams does indeed approach 1 as p increases, with a proof worth sharing.

Theorem. $$\lim_{p\to\infty} \left(\frac{1}{2^p} \left(F_{p-2} + \sum_{i=1}^{p-1} F_i 2^{p-1-i} \right) \right) = 1$$

Proof. Since there are 2^p chord diagrams when we take all possible transpositions, we now have the following inequalities. (Dropping the F_{p-2} term from the middle expression felt overly generous at first, because F_{p-2} gets large as p increases. But the move turned out so well that we might call this sacrifice, "golden.")

$$\sum_{i=1}^{p-1} F_i 2^{p-1-i} \leq \sum_{i=1}^{p-1} F_i 2^{p-1-i} + F_{p-2} \leq 2^p.$$

There is a Fibonacci formula using the Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$. It states $F_n = \dfrac{\varphi^n - (\frac{-1}{\varphi})^n}{\sqrt{5}}$ [4], which turns the left-hand sum into a difference of two geometric series, when we divide by 2^p. Taking the limit as $p \to \infty$ squeezes the ratio of null chord diagrams to 1. The interesting math happens in the left-hand limit. Here, we substitute for φ and rationalize denominators, after taking the limit.

$$\lim_{p\to\infty} \tfrac{1}{2^p} \sum_{i=1}^{p-1} F_i 2^{p-1-i} = \lim_{p\to\infty} \tfrac{1}{2\sqrt{5}} \sum_{i=1}^{p-1} \frac{\varphi^i - (\frac{-1}{\varphi})^i}{2^i} =$$
$$\frac{1}{2\sqrt{5}} \left(\frac{\varphi}{2-\varphi} + \frac{1}{2\varphi+1} \right) = \frac{1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{3-\sqrt{5}} + \frac{1}{2+\sqrt{5}} \right) =$$
$$\frac{1}{2\sqrt{5}} \left(\frac{4\sqrt{5}+8}{4} + \frac{2-\sqrt{5}}{-1} \right) = \frac{1}{2\sqrt{5}} \left(\sqrt{5}+2+\sqrt{5}-2 \right) = 1 \qquad \square$$

4. Knot theory context. We would like to take a moment to show how this paper started with knot theory. A knot is a loop in three-dimensional space, like a shoelace with the ends sealed together. The loop may weave in and around itself in all sorts of complicated ways. Telling knots apart has been, and remains, an important part of knot theory. A generalization of knots, called singular knots, has turned out to be quite useful in sorting knots. A *singular knot* is a knot which is allowed to pass through itself. A place where the singular knot intersects itself is a *singularity.*

A chord diagram represents a family of singular knots where each singularity is a chord. If we treat the knot as an oriented loop, we can draw a chord for the place where two points of the loop are treated as the same point, hence our naming the two endpoints of a chord with the same name. An isolated chord is like a pinch in a singular knot. Such a pinch is the least interesting of singularities and, in some calculations, the pinch causes the singular knot to contribute nothing. The chord diagrams with isolated chords get attention because they represent singular knots with pinches. This is also the reason why they are called null diagrams.

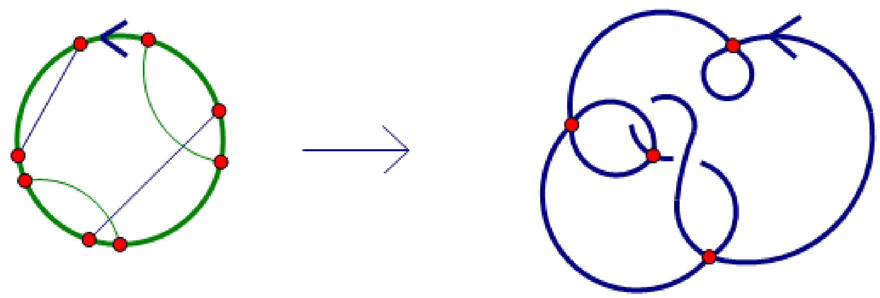

Figure 3. A chord diagram represents a singular knot.

In Figure 3, we show a singular knot which our chord diagram from Figure 1 represents, with labels suppressed. The arrow marker on the circle designates the starting point for drawing the singular knot. The two little blank sections on the singular knot indicate crossings, like overpasses on a road map. It turns out the crossings can be chosen in any way which does not change the relative order of the singularities. Traveling the two pictures will show that the isolated chord corresponds to the pinched singularity while the crossed chords require a more complicated arrangement in the singular knot.

There is a signed sum of all our one-piece diagrams with p chords which has an abbreviation as a single symbol, called a *Jacobi diagram*. Our research began with Jacobi diagrams, some of which led to the restricted diagram structure in this article. The references [1] and [2] show the details for Section 4. The interested reader should consult them for more information on the knot theoretical context.

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Jane made it through Abstract Algebra and this research, despite her muscular dystrophy. She has presented her work at Michigan State and Aquinas College's Taste Of Undergraduate Research. She enjoys a good book, lunch breaks outside and joking around with the other math students.